Knowledge from Pebbles: What Can Be Counted, and What Cannot

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THERE IS AN IDEA, powerful across the long history of formation of much of what we take to be knowledge, that the objects of our thought can best be understood as pebbles.

By way of explaining this cryptic point, let us remind you of one of Borges's last stories, "Tigres azules" (Blue tigers).[1] Its narrator, Alexander Craigie, was a Scottish philosopher who made a living teaching "occidental logic" at Lahore (modern Pakistan) circa 1900. Professor Craigie was in every way an apostle of reason, except that since his earliest childhood he had been fascinated by tigers, which even populated his dreams (already we should feel a slight tension between ways of knowing). Toward the end of 1904 Craigie read somewhere the surprising news that a blue variant of the species had been sighted. He dismissed the report as product of error or linguistic confusion, but eventually even the tigers in his dreams turned blue. Unable to resist his curiosity, he set off toward the sources of the rumor.

When he arrived at a Hindu village mentioned in some of the reports and told the villagers what he was looking for, he found that they became quite guarded, but they claimed to know of this blue tiger, and promised to help him find it. Often they led him urgently in a direction where it had purportedly just been seen, but never was it there to be found. Over time he noticed that their excursions always avoided one direction, and when he proposed to explore there he was met with consternation. That area was sacred. Any mortal who walked there would go mad or blind. Our narrator did not argue, but snuck off in the night, on the forbidden path.

The ground was sandy and full of channels. Suddenly, in one of the channels, he saw a flash of the same blue he had seen in his dreams. "The channel was full of pebbles, identical, circular, very smooth and a few centimeters in diameter" (525). They were so regular as to look artificial, like tokens or counters. He put a handful in his pocket. Back in his hut he reached into his pocket to remove a few. He felt a tickle, a tremor in his hand, and opening it saw some thirty disks, although he could have sworn that he had not taken more than ten from the channel. He did not need to count them to see that they had multiplied, so he put them in a pile and tried to count them one by one.

"This simple operation proved impossible." He would stare at any one of them, remove it with thumb and index finger, and as soon as it was alone it was (they were?) many. He closed his eyes, repeated slowly the eight definitions and seven axioms of Spinoza's *Ethics*, but "the obscene miracle" repeated itself over and over. At first he suspected he was crazy, but with time he realized that madness would have been preferable: for "if three plus one can equal two or fourteen, reason is an insanity."

Back in Lahore our logician carried out experiments, marking some pebbles with crosses, filing others, attempting to introduce some difference into their sameness by which he might distinguish them. He charted their increase and decrease, "trying to discover a law," but they changed their marks and their number in no discernible pattern. "The four operations of addition, subtraction, multiplication and division were impossible. . . . Math, I told myself, had its origins and now its end in pebbles. If Pythagoras had worked with these. . . . After a month I understood that the chaos was inescapable” (530).

Why chaos? In order to answer that question, we have to remember both what counting is, and how much of our knowledge is built upon it. First, “what is counting.”[2] Ask a mathematician and she might say: “to count a finite set is to assign to its elements, in a one-to-one manner, the numbers 1, 2, 3, . . . , *n*, without missing any one of the latter.” The task of counting a finite set is a typical math problem, in that everything is *given*, except for what we are supposed to find on the basis of the given, in this case the number of elements in set A. We say, “given a set A,” or, “given the elements of the set A,” and, “given the natural numbers 1, 2, 3, 4, and so on,” and we are supposed to find the one-to-one assignment or correspondence, and the number *n*. Having done so, we conclude, “The set A has *n* elements.”

We have counted. As mathematicians we simply assumed a very few things. We assumed the set. We assumed that there is no question as to what elements belong to the set. And we assumed that an element *x* is not changed by being counted in the set (is not affected, for example, by being placed in the set, or coming into contact with some other element within the set). A version of this last assumption we moderns call the Principle of Identity: for all x, $x = x$. Then all we needed were the most basic logical requirements for talking about natural numbers, namely: there should be a first number, say 1 (or 0); and it should always be possible to speak of “the following number,” or “follower,” or “this number plus 1.” For convenience we give these names: the follower of 1 is named 2, which can also be written as $1 + 1$; the follower of 2 is named 3, and so on for the infinity of the natural numbers. But all we really needed was the principle of identity, the one, and the possibility of speaking of the $1 + 1$.

With a twist, that is what John von Neumann did in his early work on set theory. He defined the number zero by the empty set (defined as the [unique] set ø for which: **For every *x*, *x* does NOT belong to ø**), then 1 by the set consisting only of the empty set, and so on, then 2 by the set consisting of the empty set and the set containing only the empty set; and showed how from this foundation not only the natural numbers but all of math can be constructed.[3]

The ancients were well aware of the power of these few assumptions (principle of identity and the possibility of repetition), and put it to the work of constructing not only mathematics, but the entire cosmos. We could talk here of Plato's well-known creations in the *Timaeus* and the *Epinomis*, or St. Augustine's less studied meditations on $7 + 3 = 10$ in the *Confessions* and *On the Free Choice of the Will*.[4] But our favorite example comes from the very first article of the very first Islamic "encyclopedia," compiled in the ninth century by the Brethren of Purity, describing how the One God created the cosmos:

> The Creator, exalted is His name . . . invented and innovated from the light of His unity . . . a simple substance called 'Active Intellect,' as He made two arise from one, by repetition. Then He made the Universal Soul arise from the light of the [Universal] Intellect, as He made three from the adding of one to two. Then He fashioned Prime Matter from the movement of the [Universal] Soul, as He generated four by adding one to three, and so forth.[5]

Note how math works here for monotheism: by modeling divine creation on the eternal identity of the number one and the repetitive move of the $1 + 1$, the multiplicity of the cosmos is created, while the unity of Being is maintained.

But back to pebbles. Borges put them at the origins of mathematics because Pythagoras and his followers were said by the ancients to have made their mythical discoveries by laying out *psephoi* (the Greek word means pebbles, as well as votes: a fact with considerable implications for democracy) into squares, rectangles, and triangles, thereby facilitating the remarkable combination of arithmetic and geometry that convinced them (according to Aristotle) that "all is number."[6] Why pebbles? Perhaps because under normal human timescales, temperatures, and pressures, they approximate the conditions necessary for counting: they remain identical to themselves. They do not interact with other pebbles when brought into a set (depending, that is, on the chemical composition of the pebbles: some interactions could be dramatic), nor seem to change in the act of counting.

Other things in the world are more difficult to count, whether because they are more subject to "becoming," as the philosophers would say, that is, to change in the act of counting, so that it becomes difficult to speak of the 1; or because they are subject to transformative interaction when brought together with another, so that we cannot speak of the 1 + 1. When Heraclitus wanted to mock the Pythagoreans and their pebbles, he chose what he thought was an extreme example and spoke of running waters because these seem subject to constant change: "for and on the same people who step into the same rivers, other and other waters flow" (B12).[7] How can we call those waters the "same"? How can we impose the principle of identity upon them?

Pebbles versus water: two opposing paradigms at the mythical foundations of Greek knowledge. In one corner, Heraclitus the champion of flux, for whom there is no identity that cannot be transformed into difference. In the other, Pythagoras, for whom there is no difference that cannot be reduced to an identity. (We know the move is reductive, but hope it is also heuristic.)

The conviction that we can treat things as pebbles, that we can apply the principle of identity to objects of our thought so as to make them knowable through number, has proven tremendously powerful in forming knowledge about the world: so powerful that it has conquered even Heraclitus's rushing waters. In the eighteenth century the Bernoullis, father and son, founded the field of fluid dynamics by applying Newton's calculus and laws of motion to the droplet (*guttula*) or particle (*particula*) of fluid *as if* droplet or particle were mass points.[8] That was the basic insight of Newton's calculus: that we can simplify an object by splitting it into "infinitesimals," then put those back together by means of integration, treating its objects as if they remained constant whether we collect them together or separate them. Calculus treats its objects of thought like pebbles, which is what the word *calculus* literally means in Latin.

What the Bernoullis did to water, the last century has done for human society, treating the ocean of humanity as a set of elements, and applying axiomatized theories to that set. (Think, for example, of von Neumann and Morgenstern's theory of games.) The productive power of these applications of the identity principle is enormous, so great that over millennia it has persuaded many that, in the words of Alfred North Whitehead, "of all things it is true that two and two make four."[9] Borges's pebbles are meant to help us interrogate that conviction, reminding us that where the principle of identity does not hold, 2 + 2 need not equal 4.

Spoiler alert: Borges does not let his logician descend completely into madness. After a sleepless night wandering through the city, Craigie finds himself at dawn standing before a mosque. Moved by some impulse he enters, and prays to be relieved of the burden. A blind beggar mysteriously appears and asks for the contents of Craigie's pocket. Craigie gives him the blue disks, which fall noiseless into the beggar's hands, "as if to the bottom of the sea." The beggar's response: "I do not yet know the nature of the alms you have given me, but mine to you are terrible. Yours are the days and the nights, sanity, habits, and the world" (531).

What is "terrible" about a world deprived of blue pebbles? What has it lost? Borges does not tell us directly. Neither does Newton, the inventor of calculus, who seems to share, if the words attributed to him are true, this sense of loss: "I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."[10] What objects of our thought lie in that ocean? What objects of our thought act like blue pebbles, rather than countable ones?[11]

Few questions are more important in the history of the formation of our knowledge, and answers have ranged from those who hold that everything is a countable pebble, to those (much rarer, at least among scientists) who hold that everything is a blue one,[12] and still others who compromise by applying the identity principle strictly to some realms but not others: claiming, for example, that the laws of physics do not apply to psychology.

By highlighting some of the conditions upon which the identity principle depends, we are suggesting our own answer to this question: every object of thought for which the principle of identity does not strictly hold—every object that is not purely mathematical?—is in some way, from some perspective or for the purpose of some question, a blue pebble. Whether to treat an object of our thought *as if* it is water—to bring back Heraclitus—or *as if* it is stone, as blue pebbles or as *psephoi*, that is a choice we should be free to make, depending on the question we are trying to answer, and the losses we are willing to tolerate.

What is remarkable is how often, as we work to form knowledge about ourselves, we seek to forget that freedom, rather than exercise it. Our social and behavioral sciences seem to us especially afflicted by this oblivion. It is too easy to point to the usual suspects, but we will nonetheless. In economics and political science, Game Theory, Rational Choice Theory, Revealed Preference Theory, and many other tools of the trade treat individuals as elements of a set, assume stable preferences, ceteris paribus, etc.: in short, assume the pebbliness of persons. Similarly in modern psychology and "brain science," where experimental knowledge emerges only from repeatability. (Readers searching for a critique of this limitation can still do no better than Søren Kierkegaard's devastating *Repetition: A Venture in Experimenting Psychology* of 1843.)

This willful forgetting of our freedom of methodological choice has countless consequences for our knowledge of ourselves and our societies. To take just one example, the axiomatization of our social sciences may weigh the scales in favor of individualism and against communitarianism, since in set theory and math, sets are completely determined by their constituent individuals, but the latter are unaffected by their belonging to any particular set. Dostoyevsky's famous man from the underground made a similar point much more pungently: "When they prove to you that in reality one drop of your own fat must be dearer to you than a hundred-thousand of your fellow creatures, and that this conclusion is the final solution of all so-called virtues and duties and all such prejudices and fancies, then you have just to accept it, there is no help for it, for twice two [makes four] is a law of mathematics. Just try refuting it."[13]

The costs of this amnesia are beyond our accounting, let alone the scope of this essay. But it is because those costs are so high that we disagree with Borges's conclusion, which seems to separate our freedom into two choices: either a chaotic world with blue pebbles, or a habitual one without. There are indeed those who, like Borges and Newton, divide our truths into pebbles or oceans. We think of the moving formulation of the "Sheikh of Islam" Ebü-s-Su`ūd, Chief Jurisconsult of the Ottoman Empire in the sixteenth century:

> Knowledge of Divine Truth is a limitless ocean. The Sharīah [law] is its shore. We [lawyers] are the people of the shore. The great Sufi masters are the divers in that limitless ocean. We do not argue with them.[14]

This is not, however, a separation we seek. We are not advocating that we should choose a side, nor that we should "abolish number" (*vernichte die Zahl*), to borrow a phrase of the poet Rilke's in his *Sonnets to Orpheus* (II.13). Quite the contrary. We are simply recalling our freedom to ask, about every object of our thought, what we might learn by applying the principle of identity, and what we might learn by perceiving it as blue: a freedom all the more important to exercise, the more habituated we are to a particular style of thought. If we had to phrase our exhortation in terms of poetry, we would borrow a different verse from Rilke's *Sonnets* (II.29):

> zu der stillen Erde sag: Ich rinne.
> Zu dem raschen Wasser sprich: Ich bin.
>
> [say to the rock: I flow.
> Say to the swift water course: I am.][15]

Figure 1.
Infini. By Camille Soulayrol.
Photograph by Louis Gaillard.

Notes

1. Jorge Luis Borges, "Tigres azules," in *Cuentos Completos* (Madrid: Lumen, 2011), 521–31. Quotations are cited parenthetically in the text. The story was first published in *La memoria de Shakespeare*, 1983.

2. A common enough question in philosophy. Kant, for example, discussed the work necessary to know that "7 + 5 = 12" in his *Critique of Pure Reason*, trans. Paul Guyer and Allen W. Wood (Cambridge: University of Cambridge Press, 1998), B15, 144–45. Our method and conclusions, however, are different.

3. One way to characterize von Neumann's achievement is to say that he realized that one cannot be absolutely certain that *anything in the world is a pebble and not a blue tiger*. (Remember, he couldn't simply *assume* the pebbliness of numbers, since his purpose was to provide foundations of certainty for them.)

4. There is some doubt among Plato scholars as to whether or not the *Epinomis* should be counted among the genuine dialogues, but in its treatment of number it is undoubtedly Platonic. St. Augustine: *Confessions* 6.4; *On the Free Choice of the Will* 2.15.39. St. Augustine's position is well captured in another passage from this treatise: "Enough of wisdom's being found inferior in comparison with number! They are the same. . . . The power of understanding that is present in wisdom warms those close to it, such as rational souls, whereas things that are farther away, such as physical objects, are not affected by the heat of wisdom but are suffused with the light of numbers. . . . Even if we cannot be clear whether number is in wisdom or derives from wisdom, or whether wisdom itself derives from number or is in number, or whether each can be shown to be the name of a single thing, it is certainly evident that each is true, and unchangeably true" (*On the Free Choice of the Will* 2.11.32).

5. *Epistles of the Brethren of Purity: On Arithmetic and Geometry*, trans. Nader El-Bizri (Oxford: Oxford University Press, 2013), 73.

6. Points and strokes stood for numbers in Linear B, and also in cuneiform (those archaic points remain on our dice today). According to Aristotle, the Py-

thagoreans adopted the use of pebbles as a way of visualizing numbers and arithmetic. Our use of the word "model" here is loose, and made with an awareness of the intense debate over what we might mean by "model" in mathematics. On pebbles, see, inter alia, Christoph Riedweg, *Pythagoras: His Life, Teaching, and Influence* (Ithaca, NY: Cornell University Press, 2008), 86–87.

7. Our translation of Heraclitus's fragment B12. For a slightly different translation, and for several other related fragments, see the most recent edition and translation of André Laks and Glenn W. Most, *Early Greek Philosophy: Early Ionian Thinkers*, pt. 2 (Cambridge, MA: Harvard University Press 2016), 168–71.

8. Daniel Bernoulli, *Hydrodynamica* (1738), Johann Bernoulli's *Hydraulica* (1743; the first edition falsely dated 1732). For an English translation by Tho. Carmody and H. Kobus, see Daniel Bernoulli and Johann Bernoulli, *Hydrodynamics and Hydraulics* (1968; repr., New York: Dover, 2004).

9. Alfred North Whitehead, *An Introduction to Mathematics* (New York: Henry Holt and Co., 1911), 9.

10. Sir David Brewster, *Memoirs of the Life, Writings, and Discoveries of Sir Isaac Newton* (New York: Johnson Reprint Corporation, 1965), vol. 2, 407.

11. We will not cheat by invoking other number structures with different axiomatic foundations, like integers modulo three, where two plus two is equal to one. We will talk only of natural numbers here.

12. Particularly interesting in this regard is the too often forgotten work of the chemist and philosopher of science Émile Meyerson, whose *Identité et réalité* (Paris: F. Alcan, 1908) explored the conditions for and limits of the Principle of Identity in scientific explanation. Among twentieth-century physicists, one might invoke David Bohm, or on occasion Erwin Schrödinger.

13. Fyodor Dostoyevsky, *Notes from Underground*, trans. Constance Garnett (New York: Dover Thrift, 1992), 8. Just one example: the axiomatization of our social sciences may weigh the scales in favor of individualism and against communitarianism, since in set theory and math, sets are completely determined by their constituent individuals, but the latter are unaffected by their belonging to any particular set.

14. Ebü-s-Su`ūd's *fatwā* from Shahab Ahmed's *What Is Islam? The Importance of Being Islamic* (Princeton, NJ: Princeton University Press, 2016), 288–89. Ebü-s-Su`ūd died in 1574.

15. Zu dem gebrauchten sowohl, wie zum dumpfen und stummen
Vorrat der vollen Natur, den unsäglichen Summen,
zähle dich jubelnd hinzu und vernichte die Zahl.
[To the used-up as well as to the dull, mute
stock of nature's fullness,
to the unsayable sums,
add yourself jubilantly and abolish number.]

Und wenn dich das Irdische vergaß,
zu der stillen Erde sag: Ich rinne.
Zu dem raschen Wasser sprich: Ich bin.
[And if the earthly should forget you,
say to the rock: I flow.
Say to the swift water course: I am.]